Last time

Absolute values on K: $|xy| = |x|\cdot|y|$, $|x+y| \leq |x|+|y|$

non-Archimedean $\longrightarrow$ $\leq \max(|x|,|y|)$

Archimedean: $K \hookrightarrow \mathbb{C}$

Non-Archimedean: $|x| = \alpha^{v(x)}$

$v: K^\times \longrightarrow \mathbb{R}$ valuation

$\twoheadrightarrow \mathbb{Z}$ normalised discrete valuation

$\mathcal{O} = \{x \mid v(x) \geq 0\}$ valuation ring

$\mathcal{O}^\times = \{x \mid v(x) = 0\}$ units

$\mathfrak{m} = \{x \mid v(x) > 0\}$ maximal ideal.

Ex $v_a : \mathbb{C}(t) \longrightarrow \mathbb{Z}$ order of vanishing at $a \in \mathbb{C}$

$$(t-a)^n \frac{f(t)}{g(t)} \longmapsto n$$

$\mathcal{O} = \{$ fncs with no pole at $a \}$ ← "regular at a"

$\mathcal{O}^\times = \{$ fncs with no zero or pole at $a \}$

$\mathfrak{m} = \{$ fncs f s.t $f(a) = 0 \}$

$$\mathcal{O}/\mathfrak{m} \cong \mathbb{C}$$

$$f \longmapsto f(a)$$

Suppose $v : K^\times \twoheadrightarrow \mathbb{Z}$ discrete normalised

Pick $\pi \in K$ s.t. $v(\pi) = 1$, a <u>uniformiser</u>

(p for v_p on $\mathbb{Q}$
$t - a$ for v_a on $\mathbb{C}(t)$
$\frac{1}{t}$ for v_∞ on $\mathbb{C}(t)$)

Then every $a \in K^\times$ can be written uniquely as

$$a = u \cdot \pi^n \qquad \begin{array}{l} n = v(a) \\ u \in \mathcal{O}^\times \text{ unit.} \end{array}$$

In particular,

1) $\mathfrak{m} = \{ x \in \mathcal{O} \mid v(x) \geq 1 \} = (\pi)$ principal

2) Every ideal $I \subseteq \mathcal{O}$, $I \neq \{0\}$ is (π^n) for some $n \geq 0$ [$\Rightarrow \mathcal{O}$ PID]

[namely $n = \min \{ v(x) \mid x \in I \}$]

So $\mathcal{O} \supsetneq \underset{\mathfrak{m}}{\underset{\parallel}{(\pi)}} \supsetneq \underset{\mathfrak{m}^2}{\underset{\parallel}{(\pi^2)}} \supsetneq \underset{\mathfrak{m}^3}{\underset{\parallel}{(\pi^3)}} \supsetneq \cdots$

are the non-zero ideals in $\mathcal{O}$.

§ Discrete valuation rings (DVRs)

Commutative algebra $\Rightarrow$

field of fractions

Lemma R integral domain, $K = \text{f.f.}(R)$, $R \neq K$.
The following conditions are equivalent

1) $\exists$ valuation $v : K^\times \twoheadrightarrow \mathbb{Z}$ with $\mathcal{O}_v = R$
2) $\mathcal{O}$ is a local PID.
3) $\mathcal{O}$ is Noetherian, integrally closed and has a unique non-zero prime ideal.

Such rings are called discrete valuation rings (DVRs).

Ex The ring of formal power series in 1 variable.

k field (of constants)

$$\mathcal{O} = k[[T]] = \left\{ \sum_{n=0}^{\infty} c_n T^n \,\middle|\, c_n \in k \right\}$$

formal power series.

$$K = k((T)) = \left\{ \sum_{n=n_0}^{\infty} c_n T^n \,\middle|\, c_n \in k \right\}$$

possibly < 0

formal Laurent series

$$v: K^\times \longrightarrow \mathbb{Z}$$

$$\sum_{n=n_0}^{\infty} c_n T^n \longmapsto n_0$$

$(c_{n_0} \neq 0)$

$\left(\text{e.g. } T+T^2 \overset{v}{\longmapsto} 1 \right.$
$T^2 \longrightarrow 2$
$T^{-3}+T \longrightarrow -3$
$\left. \dots \right)$

$\pi = T$ uniformiser

$\mathfrak{m} = (T)$ max. ideal

This is a DVR.

Generally, elements of DVRs may be viewed as power series:

<u>Lemma</u> $v : K^\times \longrightarrow \mathbb{Z}$ valuation, $\mathcal{O} = \mathcal{O}_v$, π uniformizer, k residue field.
$A \subseteq \mathcal{O}$ any set of representatives of k (say with $0 \in A$)

$$\left[\mathcal{O}/_{m} \xrightarrow{\sim} k \qquad \underset{\mathcal{O}}{\overset{A}{\cap}} \xleftarrow{1:1} k. \right]$$

Then every $x \in K^\times$ can be written uniquely as a convergent series

$|\text{partial sums} - x| \to 0$ in $|\cdot|$ corresponding to v

$$x = \pi^{v(x)} \sum_{n=0}^{\infty} a_n \pi^n \qquad a_n \in A, \quad a_0 \neq 0$$

Proof $x \in K^\times \Rightarrow x = \pi^{v(x)} \cdot u$ $\quad u \in \mathcal{O}^\times$ unit.

Reduce mod $\mathfrak{m}$:

$$\mathcal{O}/\mathfrak{m} \xrightarrow{\sim} k$$
$$u \longmapsto \overline{u} \neq 0$$

Choose (unique) $a_0 \in A$ s.t. $\overline{a_0} = \overline{u}$. Then

$$a_0 - u \in \mathfrak{m} \Rightarrow$$

$$u = a_0 + \pi \cdot u_1$$

Reduce u_1 mod π, lift to $a_1 \in A \Rightarrow$

$$u = a_0 + \pi a_1 + \pi^2 u_2 \quad \text{some } u_2 \in \mathcal{O}$$

and proceed.

Then $v\left(u - \sum_{i=0}^{N} a_i \pi^i\right) \geq N+1 \longrightarrow \infty$ as $N \to \infty$.

Uniqueness is clear by construction. i.e. $|\cdot| \longrightarrow 0$.

$\blacksquare$

Ex $K = \mathbb{Q}$

or $6, 10, \frac{-2}{3}$

$v = v_2$ 2-adic valuation on $\mathbb{Q}$, $\pi = 2$,

$A = \{0, 1\}$

$3, \frac{5}{7}$

{Every element in $\mathbb{Q}^\times$ is a convergent power series $\sum_{i=v_2(x)}^{\infty} a_i 2^i$, $a_i \in \{0,1\}$.

For instance,

$x = -\frac{1}{3}$ valuation 0

$\llcorner$ 1 mod 2

$= 1 + 2 \cdot \left(-\frac{2}{3}\right)$

$\llcorner$ 0 mod 2

$= 1 + 2^2\left(-\frac{1}{3}\right)$

$\llcorner$ 1 mod 2

$= 1 + 2^2 + 2^4 + 2^6 + 2^8 + \cdots$

geometric series for $\frac{1}{1-4}$

<u>Exr</u> $\sum_{n=n_0}^{\infty} a_n 2^n$, $a_n \in \{0,1\}$

converges to a number in $\mathbb{Q}$ (w.r.t. $|\cdot|_2$)

$\Leftrightarrow$ (a_n) is (eventually) periodic.

[like for decimal expansions !]

<u>Ex</u> $K = k(t)$, $v = v_0$ (order of vanishing at 0),

$\pi = t$, $A = k$ ← in this case $\mathcal{O}_v$ is a k-algebra, as opposed to the previous example.

$$x = \frac{1+t}{1-t} = 1+2t+2t^2+2t^3+\dots$$

§ Complete fields

$K, |\cdot|$ abs. value $\Rightarrow$ K metric space, $+, -, \times, \frac{1}{x}$
$d(x,y) = |x-y|$ continuous.

Def $x_n \longrightarrow x$ if $|x_n - x| \longrightarrow 0$ as $n \to \infty$.
(x_n) is Cauchy if $|x_n - x_m| \to 0$ as $n, m \to \infty$

<u>Def</u> K is <u>complete</u> if every Cauchy sequence converges. (← so {convergent seqs} = {Cauchy seqs}).

<u>Def</u> $K, |\cdot|$. The topological completion of K w.r.to $|\cdot|$ is a field $\widehat{K}$, called the <u>completion</u> of K (w.r.to $|\cdot|$).

Construction as follows:

$\mathcal{C} = \{\text{Cauchy sequences in } K\}$ ring.

$\mathcal{I} = \{\text{sequences } a_n \to 0\}$ ideal

$\hat{K} := \mathcal{C}/\mathcal{I}$ field (check)

$K \hookrightarrow \hat{K}$

$a \longmapsto (a, a, a, \ldots)$

Cauchy sequence in $\mathbb{R}$ $\Rightarrow$ converges.

$|\cdot|$ extends to $\hat{K}$ by $|(x_n)| := \lim_{n\to\infty} |x_n|$

Easy to check:

- K is dense in $\hat{K}$.
- $K = \hat{K} \iff K$ complete
- K Archimedean $\iff$ $\hat{K}$ Archimedean.
- $|\cdot|_1 \sim |\cdot|_2$ on $K \Rightarrow \hat{K}^{(1)} \cong \hat{K}^{(2)}$ as a topological field.
- $L \hookrightarrow K$ hom. of fields $\Rightarrow$ extends uniquely to $\hat{L} \hookrightarrow \hat{K}$.

<u>Ex</u> $K = \mathbb{Q}$, $|\cdot|_\infty \implies \hat{K} = \mathbb{R}$

$K = \mathbb{Q}(i)$, $|\cdot|_\infty \implies \hat{K} = \mathbb{C}$

<u>Ex</u> $K, |\cdot|$ Archimedean $\overset{\text{Ostrowski II}}{\implies}$

$$\underset{|\cdot|_\infty}{\mathbb{Q}} \hookrightarrow K \hookrightarrow \underset{|\cdot|_\infty}{\mathbb{C}}$$

Take completions $\implies$ $\mathbb{R} \hookrightarrow \hat{K} \hookrightarrow \mathbb{C}$

So the only complete Archimedean fields are $\mathbb{R}$ and $\mathbb{C}$.

$\underline{\text{Ex}}$ $K = k(t)$

$v = v_0 \longleftrightarrow |\cdot|$

$\llcorner \; v_0\left(t^n \frac{f(t)}{g(t)}\right) = n$

Here $\hat{K} = k((t))$.

Generally, C/k algebraic curve, $P \in C(k)$ non-singular

$\Rightarrow$ completion of $k(C)$ w.r.t. $|\cdot|_P$ is $\cong k((t))$.

For discrete valuations

$$v : K^\times \twoheadrightarrow \mathbb{Z}$$

recall that

$$K \hookrightarrow \left\{ \text{series } \sum_{n=n_0}^{\infty} a_n \pi^n \;\middle|\; a_n \in A \right\} \cup \{0\}$$

A: set of reps of k in $\mathcal{O}$

sequence of such series is Cauchy $\iff$ they agree to higher and higher # of terms.

$$\Rightarrow \text{RHS} = \hat{K} \quad !$$

That is,

<u>Thm</u> $v: K^\times \longrightarrow \mathbb{Z}$, π, A as above. Then

$$\hat{K} = \left\{ x = \sum_{n=n_0}^{\infty} a_n \pi^n \;\middle|\; a_n \in A \right\} \cup \{0\} \quad \text{complete field}$$
$(a_{n_0} \neq 0)$

$$\cup$$

$$\hat{\mathcal{O}} = \left\{ x = \sum_{n=0}^{\infty} a_n \pi^n \;\middle|\; a_n \in A \right\} \quad \text{DVR.}$$

§ <u>p-adic numbers</u>

<u>Def</u> $\mathbb{Q}_p$ = completion of $\mathbb{Q}$ w.r.to $|\cdot|_p$,

the <u>field of p-adic numbers</u>.

$\mathbb{Z}_p$ = closure of $\mathbb{Z}$ in $\mathbb{Q}_p$

= valuation ring of $(\mathbb{Q}_p, v_p)$,

the <u>ring of p-adic numbers</u>.

Explicitly,

$$\mathbb{Q}_p = \left\{ \sum_{n=n_0}^{\infty} a_n p^n \;\middle|\; a_n \in \{0,\dots,p-1\}, a_{n_0} \neq 0 \right\} \cup \{0\}.$$

p-adic digits

elt. of valuation n_0 (absolute value p^{-n_0})

$$\mathbb{Z}_p = \left\{ \sum_{n=0}^{\infty} a_n p^n \;\middle|\; a_n \in \{0,\dots,p-1\} \right\}$$

Rmk $\mathbb{Q}_p$ Uncountable field, $\mathbb{Q}_p \supseteq \mathbb{Q}$ [$\Rightarrow$ characteristic 0].

<u>Ex</u> In $\mathbb{Q}_2$

$$2 = 2$$
$$5 = 1 + 2^2$$

<u>Exc</u> $x \in \mathbb{N}$ $\Leftrightarrow$ has finite p-adic expansion

$$-1 = 1+2+2^2+2^3+2^4+\dots$$

← geom. series for $\frac{1}{1-2} = \sum 2^n$

$$-1/3 = 1+2^2+2^4+2^6+\dots$$

$$-2/3 = 2+2^3+2^5+2^7+\dots$$

recall: eventually periodic $\Leftrightarrow x \in \mathbb{Q}$.

$$-1/12 = 2^{-2}+1+2^2+2^4+\dots$$

$\longleftarrow \notin \mathbb{Z}_2$.

Added & multiplied like power series (with carry)

$$
\begin{array}{rrl}
 & 5 & = 1 + 2^2 \\
+ & -1/3 & = 1 + 2^2 + 2^4 + 2^6 + \dots \\
\hline
 & 4\,2/3 & = 2 + 2 \cdot 2^2 + 2^4 + 2^6 + 2^8 + \dots \\
 & & = 2 + 2^3 + 2^4 + 2^6 + 2^8 + \dots
\end{array}
$$

(for division, use geometric series).

§ Metric & topology on $\mathbb{Q}_p$

- open balls

centre $\in \mathbb{Q}_p$ radius > 0

$$\{x \in \mathbb{Q}_p \mid |x-a|_p < r\} = \{x \in \mathbb{Q}_p \mid |x-a|_p < p^{-n}\}$$

$\left(\frac{1}{p}\right)^{v_p(x-a)}$

$$= \{x \in \mathbb{Q}_p \mid v_p(x-a) \geq n\}$$

$= \{x \in \mathbb{Q}_p \mid$ p-adic digits of x and a agree up to the n^{th} one$\}$.

Ex $\mathbb{Z}_p = \{ x \in \mathbb{Q}_p \mid v(x) \overset{>-1}{\geq} 0 \}$

centre $a=0$, radius 1

$p^n\mathbb{Z}_p$, $7 + p^n\mathbb{Z}_p$

fund. system of nbds of 0 .... of 7

- closed balls = open balls.
- $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, $\mathbb{Q}$ dense in $\mathbb{Q}_p$ (Exc.)
- $\mathbb{Z}_p$ is compact totally disconnected.
- Trivia: $\mathbb{Z}_2$ is homeomorphic to the Cantor set.